Advanced Mathematics Support Programme

Multiple Equations

If $\frac{ab}{a+b} = \frac{1}{4}$ and $\frac{bc}{b+c} = \frac{1}{2}$ and $\frac{ac}{a+c} = \frac{1}{8}$ find a, b and c

Rearrange the equations to get the following 3 equations

$$4ab = a + b \qquad (1)$$

$$2bc = b + c \qquad (2)$$

$$8ac = a + c \qquad (3)$$

Rearrange (1) to make b the subject

$$a = 4ab - b$$
$$b(4a - 1) = a$$
$$b = \frac{a}{4a - 1}$$

Rearrange (3) to make c the subject

$$a = 8ac - c$$
$$c(8a - 1) = a$$
$$c = \frac{a}{8a - 1}$$

Substitute expressions for b and c into equation (2) $\mathbf{2bc = b + c}$ giving

$$2\left(\frac{a}{4a-1}\right)\left(\frac{a}{8a-1}\right) = \frac{a}{4a-1} + \frac{a}{8a-1}$$

$$\frac{2a^2}{(4a-1)(8a-1)} = \left(\frac{a(8a-1)}{(4a-1)(8a-1)}\right) + \left(\frac{a(4a-1)}{(4a-1)(8a-1)}\right)$$

$$\frac{2a^2}{(4a-1)(8a-1)} = \frac{8a^2 - a + 4a^2 - a}{(4a-1)(8a-1)}$$

$$\frac{2a^2}{(4a-1)(8a-1)} = \frac{12a^2 - 2a}{(4a-1)(8a-1)}$$

Equating the numerators only as denominators are the same expressions gives

$$2a^2 = 12a^2 - 2a$$

$$10a^2 - 2a = 0$$
$$2a(5a-1) = 0$$
$$a = 0 \quad or\ a = \frac{1}{5}$$

$a = \frac{1}{5}$ is the only possible solution

*Can you explain why that is the case? Why can't $a = 0$?

Substituting this value into (1) gives

$$4ab - a = b$$
$$4 \times \frac{1}{5}b - \frac{1}{5} = b$$
$$\frac{4}{5}b - b = \frac{1}{5}$$
$$-\frac{1}{5}b = \frac{1}{5}$$
$$b = -1$$

Substituting $a = \frac{1}{5}$ into (3) gives

$$8ac - a = c$$
$$8 \times \frac{1}{5} \times c - \frac{1}{5} = c$$

$$\frac{8}{5}c - c = \frac{1}{5}$$
$$\frac{3}{5}c = \frac{1}{5}$$
$$c = \frac{1}{3}$$

Giving the solution $\boldsymbol{a = \frac{1}{5}}$ $\quad \boldsymbol{b = -1}$ **and** $\boldsymbol{c = \frac{1}{3}}$

*To explain why a can't be $= 0$ consider the original statements $\frac{ab}{a+b} = \frac{1}{4}$ and $\frac{ac}{a+c} = \frac{1}{8}$

They would both have a numerator of 0

0÷by 'anything' is 0 (except for $0 \div 0 = undefined$)

Therefore the statements would not be correct