Advanced Mathematics Support Programme®

AMSP Professional development videos: Problem solving

Aims of the session
■ To encourage teachers to reflect on their own practice and explore ways of incorporating approaches to problem solving into their lessons. ■ To provide teachers with the opportunity to use and create a bank of resources that can be trialled in future lessons.
Resources needed
■ A projector for showing the videos. ■ Access to a bank of examination questions. ■ Tasks A and B and sample questions.

Introduction

Look at Task A in pairs. Work together to solve the problem, noting down any key questions or observations that arise out of your discussions. Once everyone has solved the problem compare the notes that each pair made.

Although the question was not stated, an experienced problem solver will work out that they are required to find the value of each shape and then sequentially work through finding each one to obtain the solution.

Teachers are necessarily experienced problem solvers! A common reason why students struggle with problem solving tasks is that they lack sufficient practice in internally asking themselves the sort of questions that allow them to progress through a problem. “I don’t know where to start!” and “What is it even asking?” are typical responses, yet once the problem is begun students will often realise that they possess the necessary mathematical skills.

Watch the first video clip in the Problem solving series, which shows a teacher providing his class with a task that forces them to externalise the questions that they need to ask to access the problem.

After watching the video you might like to discuss the following questions:

- How was the task effective in engaging the students with the problem?
- How can we ensure that all students take part, rather than a vocal minority?
- What might be the next step in helping students to *internally* ask the necessary questions to solve a problem?

Individual/pair work

Select an extended question from a past paper, ideally one broken down into intermediate steps. Reword the question so that no intermediate steps are provided, and remove some key necessary information from the question. Once this is done, list the questions that you would anticipate students asking in order to access and progress through the problem. Present your problem to a partner and compare the questions asked with the ones you anticipated. If no past paper questions are available there are two example questions provided.

Alternative approach

Watch the second video, which demonstrates how group work can be used to develop problem solving skills.

Individual/pair work

Attempt Task B in pairs or small groups and discuss the opportunities and challenges that this sort of task provides. Attempt to identify areas of the syllabus that might lend themselves well to this sort of activity. If time allows, attempt to produce a task using this structure. Possible ideas could include:

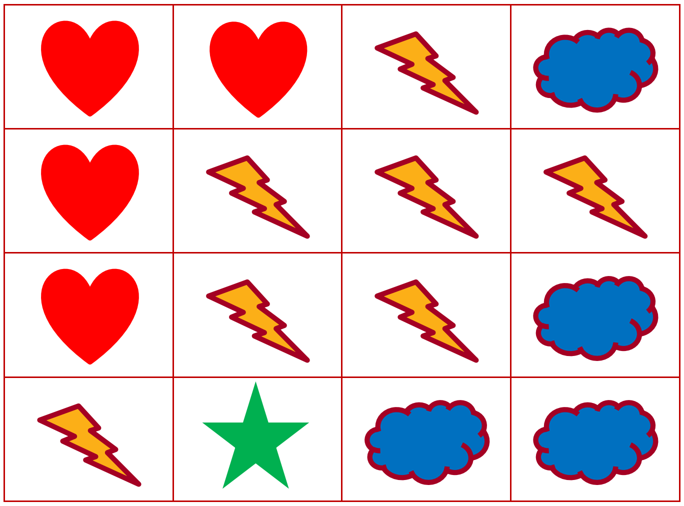
- Finding the combined sum of specified terms in different sequences (cards might include “Take the third term from sequence B” or “Sequence D has common ratio 3”).
- Performing a set of graphical transformations in a specified order, leading to a final point (cards might include “The fourth transformation is a stretch, and follows a translation” or “The second reflection is in the x axis”).
- Finding the area or volume of a complicated shape with lengths that involve surd calculations (cards might include “The shape consists of three triangles joined to a square and a rectangle” or “The diagonal of the square has length $7\sqrt{6}$”).

Next steps

The other videos in this section provide further ideas for how to develop students’ problem solving skills. In addition, MEI has produced a ***problem solving guide***, discussing many of these ideas in depth and providing further worked examples.

Task A

Solve the puzzle



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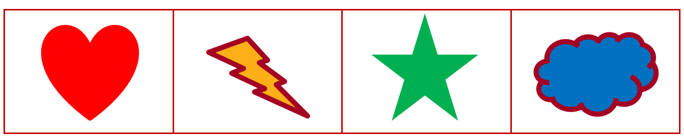


?

Solution

The answer is 23.

The task involves working out the value of each shape from the total given at the end of each row. There are different approaches that can be taken, and one is given below.

By comparing rows 1 and 3 we can ascertain that a heart must be worth 3 more than a lightning bolt. Applying this to row 2, we find a lightning bolt to be worth 4 and therefore a heart to be worth 7. Row 1 or 3 can then be used to find that a cloud is worth 9, with row 4 then telling us that a star must be worth 3.

Task B

Share out the cards within a pair or group.

Each person may verbally share the information on their cards, but must not show their cards to the others.

Once completed, work out which cards are not needed to complete the problem.

The x axis on the map goes from 0 to 5	y is greater than $2^{-x} + 1$	The y axis on the map goes from 0 to 5
The treasure is contained within the circle $x^2 + y^2 = 10y$	The treasure is buried at a point with integer coordinates	x is less than or equal to 4
The task is to find the buried treasure	The treasure map is based on a Cartesian coordinate grid	The treasure lies between the curve $y = 6x - x^2 - 5$ and the x axis
The x and y coordinates are not the same	The y coordinate is less than the x coordinate	x and y are both positive

Solution

The treasure is buried at the point (3,2).

Not all cards are needed to locate this point. The following cards are sufficient for narrowing down the location:

- The treasure is contained within the circle $x^2 + y^2 = 10y$
- y is greater than $2^{-x} + 1$
- The treasure lies between the curve $y = 6x - x^2 - 5$ and the x axis
- The y coordinate is less than the x coordinate
- The treasure is buried at a point with integer coordinates

The following two cards define the problem, but it could be argued that they can be inferred from the others:

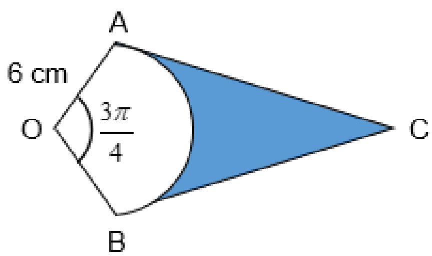
- The task is to find the buried treasure
- The treasure map is based on a Cartesian coordinate grid

The remaining cards may help with visualisation, but are superfluous.

Sample question 1

Topic links: Trigonometry and radians

Original problem:

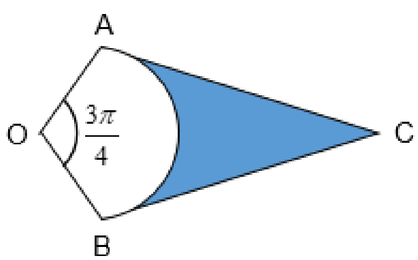


OAB is a sector of a circle with radius 6 cm.

The lines AC and BC are tangents from this circle.

Find the shaded area ABC, giving your answer to three significant figures.

Problem with key information missing:



Find the shaded area ABC.

Questions that students will need answers to:

Is AOB a sector of a circle?

What is its radius?

Are AC and BC tangents to the circle/equal/perpendicular to OA and OB?

Is an exact or rounded answer required?

These questions are sufficient to establish the problem – students may need to ask further questions to successfully complete the solution.

Solution:

The shaded area can be found by subtracting the area of the sector OAB from the area of the kite OACB.

The kite is formed of two equal right-angled triangles and therefore has an area equal to a rectangle with lengths OA and AC. Due to the symmetry of the shape, angle AOC is half the angle AOB, and so we have

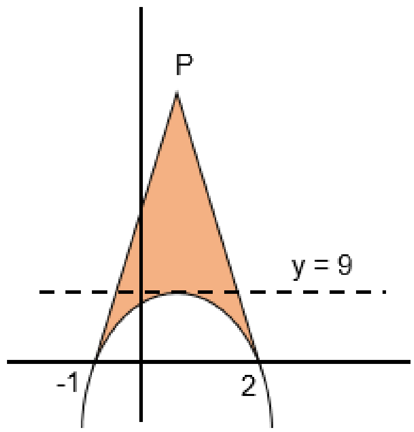
$$AC = 6\tan\frac{3\pi}{8}$$

Therefore the required area is $6 \times 6\tan\frac{3\pi}{8} - \frac{1}{2} \times 6^2 \times \frac{3\pi}{4} = 44.5 \text{ cm}^2$

Sample question 2

Topic links: Co-ordinate geometry and integration

Original problem:

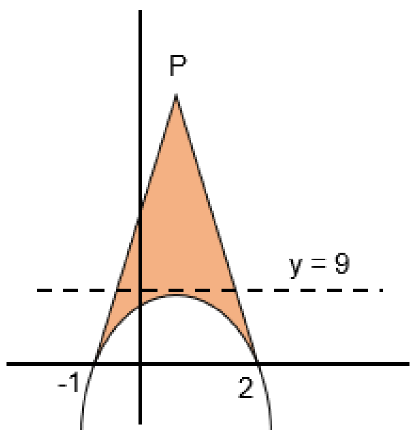


The diagram shows a quadratic function which has a maximum value of $y = 9$.

Tangents to the curve are drawn at $x = -1$ and $x = 2$. These tangents meet at the point P.

Find the shaded area enclosed by the curve and the two tangents at P.

Problem with key information missing:



Find the shaded area.

Questions that students will need answers to:

Is the curve quadratic or some other type of function?

Does the dashed line indicate the maximum point?

Are the diagonal lines tangents to the curve?

These questions are sufficient to establish the problem – students may need to ask further questions to successfully complete the solution.

Solution:

Given that the curve represents a quadratic function it must have form $y = -k(x + 1)(x - 2)$.

As it is symmetrical, the maximum value will occur when x is halfway between -1 and 2. We therefore have:

$$y = -k(x+1)(x-2)$$
$$9 = -k(\frac{1}{2}+1)(\frac{1}{2}-2)$$
$$9 = \frac{9k}{4}$$
$$k = 4$$

We can now differentiate to find the equations of the two tangents:

$$y = -4(x+1)(x-2)$$
$$y = -4x^2 + 4x + 8$$
$$\frac{dy}{dx} = -8x + 4$$

Substituting x = -1 and 2 respectively produces gradients of 12 and -12, leading to the lines

$y = 12x + 12$ and $y = -12x + 24$

These meet when x = 0.5 and y = 18.

We can now find the required area by subtracting the integral of the curve between -1 and 2 from the area of the triangle formed by the two tangents and the x axis:

$$Area = \frac{1}{2} \times 3 \times 18 - \int_{-1}^{2} -4x^2 + 4x + 8 dx$$
$$= 27 - \left[-\frac{4x^3}{3} + 2x^2 + 8x \right]_{-1}^{2}$$
$$= 27 - (\frac{40}{3} + \frac{14}{3})$$
$$= 27 - 18$$
$$= 9$$